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FUZZY PRE-γ-COMPACT, FUZZY PRE-γ-CONNECTED AND FUZZY PRE-γ-CLOSED SPACES

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Abstract: Compactness and connectedness play a crucial role in Topology. In this paper, we introduce concepts of fuzzy pre-γ-compact, fuzzy pre-γ-connected and fuzzy pre-γ-closed spaces by using concepts of fuzzy pre-γ-open sets. Then we study their properties and compare them.

Keywords and Phrases: Fuzzy pre-γ-open sets, fuzzy pre-γ-compact, fuzzy pre-γ-connected, fuzzy pre-γ-closed spaces.

2020 Mathematics Subject Classification: 54A05, 54A40, 54D05, 54D30.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh in his paper [12]. By using the concept of fuzzy sets, Chang [1] introduced the idea of fuzzy topological space. The notion of fuzzy sets has been used by many researchers to several branches of Mathematics. Kasahara [7] defined the notion of an operation γ on a topological space. Kalitha and Das [6] introduced and investigated the operation γ on fuzzy topological spaces. The notion of pre-γ-open sets in general topological spaces was defined by Ibrahim [5]. Recently Sivashanmugaraja and Vadivel [11] introduced the notion of pre-γ-open fuzzy sets in fuzzy topological spaces. The aim of this paper is devoted to introduce and investigate the notion of pre-γ-compact, pre-γ-connected and pre-γ-closed spaces in fuzzy setting. Also we establish some basic theorems about it.

2. Preliminaries

Throughout this paper $(X,\ \tau_X)$ or simply X always mean a fuzzy topological space (fts, for short). The interior, the closure and complement of a fuzzy set $A \in I^X$ will be denoted by $int(A)$, $cl(A)$ and A^c respectively. By $\underline{0}$ and $\underline{1}$ we mean the constant fuzzy sets taking on the values 0 and 1 respectively. Now we recall some of the basic definitions.

Definition 2.1. [6] *Let $(X,\ \tau_X)$ be a fuzzy topological space. A fuzzy operation γ on the topology τ_X is a mapping from τ into set I^X such that $\lambda \subseteq \gamma(\lambda),\ \forall\ \lambda \in \tau_X$ where $\gamma(\lambda)$ denotes the value of γ at V. The mapping defined as $\gamma(\lambda) = \lambda,\ \gamma(\lambda) = cl(\lambda),\ \gamma(\lambda) = int(cl(\lambda))$., etc are examples of fuzzy operations.*

Definition 2.2. [6] *A fuzzy subset λ of (X, τ_X) is called a fuzzy γ-open, if $\forall$ ${p_x}^\lambda\ q\ \lambda$, $\exists$ a $\mu \in \tau$ and ${p_x}^\lambda\ q\ \mu$ such that $\gamma(\mu) \leq \lambda$. τ_γ denotes the set of all γ-open fuzzy sets. Clearly we have $\tau_\gamma \subseteq \tau_X$.*

Definition 2.3. *A fuzzy set λ of a fts X is called fuzzy pre-γ-open* [11] *if $\lambda \leq \tau_\gamma$-$int(cl(\lambda))$. A fuzzy set λ of a fts X is called fuzzy pre-γ-closed* [9] *iff its complement is fuzzy pre-γ-open. The family of all pre-γ-open and pre-γ-closed fuzzy sets are denoted by $FP_\gamma O(X)$ and $FP_\gamma C(X)$ respectively.*

Definition 2.4. [9] *Let λ be a fuzzy subset of (X, τ). Then:*

(i) The union of all fuzzy pre-γ-open sets contained in λ is called the fuzzy pre-γ-interior of λ, denoted by $pint_\gamma(\lambda)$.
i.e., $pint_\gamma(\lambda) = \vee\{\mu \leq \lambda : \mu \in FP_\gamma O(X)\}$.

(ii) The intersection of all fuzzy pre-γ-closed sets containing λ is called the fuzzy pre-γ-closure of λ, denoted by $pcl_\gamma(\lambda)$.
i.e., $pcl_\gamma(\lambda) = \wedge\{\mu \geq \lambda : \mu \in FP_\gamma C(X)\}$.

Definition 2.5. [4] *A collection of fuzzy subsets $\mathcal{P}$ of a fts $(X,\ \tau)$ is called to form a fuzzy filterbases iff for every finite collection $\{\mathcal{Q}_i : i = 1, 2, , ..., n\}$, $\bigwedge_{i=1}^{n} \mathcal{Q}_i \neq \underline{0}$.*

Definition 2.6. [1] *A mapping $f : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is called fuzzy continuous, if $f^{-1}(\mu)$ is an open fuzzy set of X, $\forall$ open fuzzy set μ of Y.*

Definition 2.7. [10] *A mapping $f : (X,\ \tau) \to (Y,\ \sigma)$ is said to be fuzzy pre-γ-irresolute (or pre*-γ-continuous), if $f^{-1}(\lambda)$ is pre-γ-open fuzzy set of X, $\forall$ pre-γ-open fuzzy set λ of Y.*

Definition 2.8. [8] *A mapping $f : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is called fuzzy pre*-γ-open, if the image of each pre-γ-open fuzzy set of $(X,\ \tau_X)$ is pre-γ-open fuzzy set in*

$(Y,\ \tau_Y)$.

3. Fuzzy Pre-γ-Compact Spaces

In this section, we introduce notions of fuzzy pre-γ-compact, fuzzy pre-γ-compact relative to X, fuzzy locally pre-γ-compact and discuss some fundamental theorems about it.

Definition 3.1. *Let $(X,\ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. A collection $\mathcal{A}$ of pre-γ-open fuzzy subsets of a fts X is said to pre-γ-open cover X or to be a pre-γ-open covering of X, if $\bigvee\limits_{B\in\mathcal{A}} B = \underline{1}$.*

Definition 3.2. *Let $(X,\ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. Then X is called fuzzy pre-γ-compact if each pre-γ-open covering $\mathcal{A}$ of X contains a finite sub collection that also covers X.*

Example 3.1. Any fuzzy topological space $(X,\ \tau_X)$ containing only finitely many pre-γ-open fuzzy sets is necessarily fuzzy pre-γ-compact, since in this case each pre-γ-open fuzzy sets covering of X is finite.

Definition 3.3. *Let $(X,\ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. A fuzzy set λ in X is called fuzzy pre-γ-compact relative to X iff $\forall$ collection $\mathcal{A}$ of pre-γ-open fuzzy sets such that $(\bigvee\limits_{\mu\in\mathcal{A}} \mu) \geq \lambda(x)$, there is a finite subcollection $\mathcal{B}$ of $\mathcal{A}$ such that $(\bigvee\limits_{\mu\in\mathcal{B}} \mu) \geq \lambda(x)$, $\forall\ x \in S(\lambda)$.*

Theorem 3.1. *A fts X is fuzzy pre-γ-compact iff each collection $\{\mathcal{C}_i : i \in J\}$ of pre-γ-closed fuzzy sets of X having the finite intersection property, $(\bigwedge\limits_{i\in J} \mathcal{C}_i) \neq \underline{0}$.*

Proof. Let X be a fuzzy pre-γ-compact and the collection $\{\mathcal{C}_i : i \in J\}$ of pre-γ-closed fuzzy sets having the finite intersection property. Assume that $(\bigwedge\limits_{i\in J} \mathcal{C}_i) = \underline{0}$. Then $(\bigvee\limits_{i\in J} \overline{\mathcal{C}_i}) = \underline{1}$. By hypothesis $\{\overline{\mathcal{C}_i} : i \in J\}$ is a collection of pre-γ-open fuzzy sets covering X, then by definition of pre-γ-compactness of X, it follows that then $\exists$ a finite subset I of J such that $(\bigvee\limits_{i\in I} \overline{\mathcal{C}_i}) = \underline{1}$. Therefore $(\bigwedge\limits_{i\in I} \mathcal{C}_i) = \underline{0}$, which is a contradiction to our assumption. Hence $(\bigwedge\limits_{i\in J} \mathcal{C}_i) \neq \underline{0}$.

Conversely, let the collection $\{\mathcal{C}_i : i \in J\}$ of pre-γ-open fuzzy sets covering X. Assume that for each finite fuzzy subset I of J , we have $(\bigvee\limits_{i\in I} \mathcal{C}_i) \neq \underline{1}$. Therefore $(\bigwedge\limits_{i\in I} (\overline{\mathcal{C}_i})) \neq \underline{0}$. Thus $\{\overline{\mathcal{C}_i} : i \in J\}$ satisfies the finite intersection property. Therefore by hypothesis, we obtain $(\bigwedge\limits_{i\in J} \overline{\mathcal{C}}_i) \neq \underline{0}$. which implies $(\bigvee\limits_{i\in I} \mathcal{C}_i) \neq \underline{1}$. This is a contradiction to our assumption $\{\mathcal{C}_i : i \in J\}$ is a pre-γ-open fuzzy set cover of X. Hence X is fuzzy pre-γ-compact.

Theorem 3.2. *A fts X is fuzzy pre-γ-compact iff each fuzzy filterbases $\mathcal{P}$ in X, $(\bigwedge_{H\in\mathcal{P}} pcl_\gamma(H)) \neq \underline{0}$.*

Proof. Let $\mathcal{A}$ be a fuzzy pre-γ-open cover which has no finite sub-cover in X. Then $\forall$ finite sub collection of $\{\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n\}$ of $\mathcal{A}$, $\exists$ $x \in X$ such that $\mathcal{B}_i(x) < 1$, $\forall$ $i = 1, 2, ..., n$. Therefore $\overline{B}_i(x) \geq 0$, so that $(\bigwedge_{i=1}^{n} \overline{\mathcal{A}}_i(x)) \neq \underline{0}$. Hence $\{\overline{\mathcal{B}}_i(x) : \mathcal{B}_i \in \mathcal{A}\}$ forms a filterbases in X. Since $\mathcal{A}$ is pre-γ-open fuzzy set cover of X, $(\bigvee_{\mathcal{B}_i\in\mathcal{A}} pcl_\gamma(\mathcal{B}_i))(x) = (\bigvee_{\mathcal{B}_i\in\mathcal{A}} \mathcal{B}_i)(x) = \underline{1}$, $\forall$ $x \in X$ and thus $\bigwedge_{\mathcal{B}_i\in\mathcal{A}} pcl_\gamma(\overline{\mathcal{B}}_i)(x) = \bigwedge_{\mathcal{B}_i\in\mathcal{A}} \overline{\mathcal{B}}_i(x) = \underline{0}$, which is a contradiction. Then each pre-γ-open fuzzy set cover of X has a finite subcover and thus X is fuzzy pre-γ-compact.

Conversely, assume that $\exists$ a filterbases $\mathcal{P}$ in X, such that $\bigwedge_{\mathcal{H}\in\mathcal{P}} pcl_\gamma(\mathcal{H}) = \underline{0}$, so that $(\bigvee_{\mathcal{H}\in\mathcal{P}} \overline{pcl_\gamma(\mathcal{H})})(x) = \underline{1}$, $\forall$ $x \in X$ and thus $\mathcal{A} = \{\overline{pcl_\gamma(\mathcal{H})} : \mathcal{H} \in \mathcal{A}\}$ is a fuzzy pre-γ-open fuzzy set cover of X. Since X is fuzzy pre-γ-compact, we obtain $\mathcal{P}$ has a finite subcover. Therefore $(\bigvee_{i=1}^{n} \overline{pcl_\gamma(\mathcal{H}_i)})$ $(x) = \underline{1}$ and thus $(\bigvee_{i=1}^{n} \overline{(\mathcal{H}_i)})(x) = \underline{1}$, so that $\bigwedge_{i=1}^{n}(\mathcal{H}_i) = \underline{0}$, which is a contradiction. Thus $\bigwedge_{\mathcal{H}\in\mathcal{P}} pcl_\gamma(H) \neq \underline{0}$, $\forall$ filterbases $\mathcal{P}$.

Theorem 3.3. *A fuzzy set λ in a fts X is fuzzy pre-γ-compact relative to X iff each filterbase $\mathcal{P}$ such that each finite members of $\mathcal{P}$ is quasi coincident with C, $(\bigwedge_{H\in\mathcal{P}} pcl_\gamma(H)) \wedge C \neq \underline{0}$.*

Proof. If possible assume that λ is not fuzzy pre-γ-compact relative to X, then $\exists$ a pre-γ-open fuzzy set $\mathcal{A}$ covering of λ such that no finite subcover $\mathcal{B}$. Then $(\bigvee_{B_i\in\mathcal{B}} B_i)(x) < \lambda(x)$, for some $x \in S(\lambda)$, so that $(\bigwedge_{B_i\in\mathcal{B}} B_i(x) > \overline{\lambda}(x) \geq \underline{0}$ and thus $\mathcal{P} = \{\overline{B_i}(x) : B_i \in \mathcal{A}\}$ forms a filterbases and $(\bigwedge_{B_i\in\mathcal{B}} \overline{B_i})\ q\ \lambda$. Since, $(\bigwedge_{B_i\in\mathcal{B}} pcl_\gamma(B_i)) \bigwedge \lambda \neq \underline{0}$, we obtain $(\bigwedge_{B_i\in\mathcal{B}} \overline{B_i}) \bigwedge \lambda \neq \underline{0}$. Then for some $x \in S(\lambda)$, $(\bigwedge_{B_i\in\mathcal{A}} \overline{B_i})(x) > \underline{0}$, which gives $(\bigvee_{B_i\in\mathcal{A}} B_i)(x) < \underline{1}$. This is a contradiction. Thus λ is a fuzzy pre-γ-compact relative to X.

Conversely, assume that $\exists$ a filterbases $\mathcal{P}$ such that each finite members of $\mathcal{P}$ is quasi coincident with λ and $(\bigwedge_{H\in\mathcal{P}} pcl_\gamma(H)) \bigwedge \lambda \neq \underline{0}$. Then $\forall$ $x \in S(\lambda)$,

$(\bigwedge_{G\in\Gamma} pcl_\gamma(G))(x) \neq \underline{0}$ and thus $(\bigvee_{H\in\mathcal{P}} \overline{pcl_\gamma(H)})(x) = \underline{1}$, $\forall\ x \in S(\lambda)$. Hence $\mathcal{A} = \{\overline{pcl_\gamma(H)} : H \in \mathcal{P}\}$ is a pre-γ-open fuzzy set cover of λ. Since λ is fuzzy pre-γ-compact relative to X, $\exists$ a finite subcover, say $\{\overline{pcl_\gamma(H_i)} : i = 1, 2, \ldots, n\}$ such that $(\bigvee_{i=1}^{n} \overline{pcl_\gamma(H_i)}(x) \geq \lambda(x)\ \forall\ x \in S(\lambda)$. Thus $(\bigwedge_{i=1}^{n} pcl_\gamma(H_i))(x) \leq \overline{\lambda}(x)\ \forall\ x \in S(\lambda)$, so that $(\bigwedge_{i=1}^{n} pcl_\gamma(H_i))\ \overline{q}\ \lambda$ which is a contradiction. Thus $\forall$ filterbases $\mathcal{P}$, each finite member of $\mathcal{P}$ is quasi-coincident with λ. Hence $(\bigwedge_{H\in\mathcal{P}} pcl_\gamma(H)) \bigwedge \lambda \neq \underline{0}$.

Theorem 3.4. *Each pre-γ-closed fuzzy subset of fuzzy pre-γ-compact space is fuzzy pre-γ-compact relative to X.*

Proof. Assume that $\mathcal{P}$ be a fuzzy filterbases in $(X,\ \tau_X)$ such that $\mu\ q\ (\bigwedge\{\lambda : \lambda \in \mathcal{Q}\})$ holds $\forall$ finite sub collection $\mathcal{Q}$ of $\mathcal{P}$ and pre-γ-closed fuzzy subset μ. Let $\lambda^* = \{\mu\} \cup \lambda$. For any finite sub collection $\mathcal{Q}^*$ of $\mathcal{P}^*$, if $\mu \notin \mathcal{Q}^*$, then $\bigwedge \mathcal{Q}^* \neq \underline{0}$. If $\mu \in \mathcal{Q}^*$ and since $\mu\ q\ (\bigwedge\{\lambda : \lambda \in \mathcal{Q}^* - \mu\})$, we obtain $\bigwedge \mathcal{Q}^* \neq \underline{0}$. Thus $\mathcal{Q}^*$ is a fuzzy filterbases in X. Since X is fuzzy pre-γ-compact, we obtain $(\bigwedge_{\lambda\in\mathcal{P}^*} pcl_\gamma(\lambda)) \neq \underline{0}$, such that $(\bigwedge_{\lambda\in\mathcal{P}} pcl_\gamma(\lambda)) \bigwedge \mu = pcl_\gamma(\lambda) \bigwedge pcl_\gamma(\mu) \neq \underline{0}$. Hence by Theorem 3.3, μ is fuzzy pre-γ-compact relative to X.

Theorem 3.5. *The image of a fuzzy pre-γ-compact space under a fuzzy pre*-γ-continuous mapping is fuzzy pre-γ-compact.*

Proof. Obvious.

Theorem 3.6. *If a mapping $f : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is fuzzy pre*-γ-continuous and η is fuzzy pre-γ-compact relative to X, then $f(\eta)$ is fuzzy pre-γ-compact.*

Proof. Let $\{B_\mu : \mu \in \mathcal{P}\}$ be a pre-γ-open fuzzy set cover of $S(f(\eta))$ in Y. For x in $S(\eta)$, $f(x) \in f(S(\eta))$. Since f is fuzzy pre*-γ-continuous, $\{f^{-1}(B_\mu) : \mu \in \mathcal{P}\}$ is pre-γ-open fuzzy set cover of $S(\eta)$ in X. Since η is fuzzy pre-γ-compact relative to X, there is a finite subcollection $\{f^{-1}(B_\mu) : \mu = 1, 2, ..., n\}$ such that $S(\eta) \leq \bigwedge_{\mu=1}^{n} f^{-1}(B_\mu) = f^{-1}(B_\mu)$. Thus $S(f(\eta)) = f(S(\eta)) \leq f(f^{-1}(B_\mu)) \leq \bigwedge_{\mu=1}^{n} f(B_\mu)$. Hence $f(\eta)$ is fuzzy pre-γ-compact relative to Y.

Definition 3.4. *Let $(X,\ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. Then X is called fuzzy locally pre-γ-compact iff for each fuzzy point x_α in X, $\exists$ μ such that*

(i) $x_\alpha \in \mu$;

(ii) μ *is fuzzy pre-γ-compact.*

Remark 3.1. *Every fuzzy pre-γ-compact is fuzzy locally pre-γ-compact.*

The converse of the above Remark 3.1 need not be true as shown in the following example.

Example 3.2. Let $X = I = [0,1]$ and consider the following fuzzy sets $A_1(x) = 1.30/\sqrt{2}$, $A_2(x) = 1.31/\sqrt{2}$, $A_3(x) = 1.32/\sqrt{2}$, ..., $\forall\ x \in I$. Let $\tau = \{A_i : i \in N^+\} \cup \{\underline{0}, \underline{1}\}$ It is clear that τ is fuzzy topology on X. Define an operation γ on τ by $\gamma(\lambda) = \lambda$. Now the space X is fuzzy locally pre-γ-compact, since the whole space X satisfies the necessary condition. But the space X is not fuzzy pre-γ-compact, since X has no finite fuzzy pre-γ-open subcover.

Theorem 3.7. *Let $(X,\ \tau_X)$ be a locally pre-γ-compact fts and $(Y,\ \tau_Y)$ be a fts. If a fuzzy continuous mapping $f : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is fuzzy pre*-γ-open, then Y is fuzzy locally pre-γ-compact.*

Proof. Suppose x_α be a fuzzy point in Y with support x_1 and the value x. Then x_β is a fuzzy point in X with support y_1 and the value x. Now $y_1 \in f^{-1}(x_1)$. Therefore $f(x_\beta) = x_\alpha$. Since x_β is a fuzzy point in X and X is fuzzy locally pre-γ-compact, by definition there exists an element $\lambda \in \tau_X$ such that $x_\beta \in \lambda$ and λ is fuzzy pre-γ-compact. Now $\lambda \in \tau_X$ and f is fuzzy pre*-γ-open map, thus $f(\lambda) \in \tau_Y$ and $x_\alpha \in f(\lambda)$, also $f(\lambda)$ is fuzzy pre-γ-compact in Y. Hence for a fuzzy point $q \in Y$ there exist a member $f(\lambda) \in \tau_Y$ such that $x_\alpha \in f(\lambda)$ and $f(\lambda)$ is fuzzy pre-γ-compact. Thus $(Y,\ \tau_Y)$ is fuzzy locally pre-γ-compact.

4. Fuzzy Pre-γ-Closed Spaces

In this section, we introduce notions of fuzzy pre-γ-closed, fuzzy pre-γ-closed relative to X. We also prove some theorems.

Definition 4.1. *Let $(X,\ \tau_X)$ be a fts and γ be an fuzzy operation on τ_X. Then X is called fuzzy pre-γ-closed iff $\forall$ collection $\mathcal{A}$ of pre-γ-open fuzzy sets such that $\bigvee\limits_{\lambda\in\mathcal{A}} \lambda = \underline{1}$, there is a finite subcollection $\mathcal{B}$ of $\mathcal{A}$ such that $(\bigvee\limits_{\lambda\in\mathcal{B}} pcl_\gamma(\lambda))(x) = \underline{1}$, $\forall$ $x \in X$.*

Definition 4.2. *Let $(X,\ \tau_X)$ be a fts and γ be an fuzzy operation on τ_X. A fuzzy set λ in X is called fuzzy pre-γ-closed relative to X iff $\forall$ collection $\mathcal{A}$ of pre-γ-open fuzzy sets such that $\bigvee\limits_{\mu\in\mathcal{A}} \mu = \lambda$, there is a finite subcollection $\mathcal{B}$ of $\mathcal{A}$ such that $\bigvee\limits_{\mu\in\mathcal{B}} pcl_\gamma(\mu)(x) = \lambda(x)$, $\forall\ x \in S(\lambda)$.*

Remark 4.1. *Every fuzzy pre-γ-compact space is fuzzy pre-γ-closed, but the converse may not be true as shown in the below example.*

Example 4.1. Let $X(\neq \underline{0})$ be a set and $A_m(x) = 1 - \frac{1}{m}$, $\forall\ x \in X$ and m be a positive natural number. The collection $\{A_m : m$ is a positive natural number$\}$ is a base for a fuzzy topology on X. Define a fuzzy operation γ on the fuzzy topology as $\gamma(\lambda) = \lambda$, $\forall$ open fuzzy sets. The collection $\{A_m : m$ is a positive natural number$\}$ is a pre-γ-open fuzzy set cover of X. On the other hand, we obtain $pcl_\gamma(\lambda) = \underline{1}$. Thus the fts X is fuzzy pre-γ-closed but not fuzzy pre-γ-compact. (see [2])

Theorem 4.1. *A fts X is fuzzy pre-γ-closed iff for every fuzzy pre-γ-open filter-bases $\mathcal{P}$ in X, $(\bigwedge_{H\in\mathcal{P}} pcl_\gamma(H)) \neq \underline{0}$.*

Proof. Let $\mathcal{A}$ be a pre-γ-open fuzzy set cover of X and let for each finite collection $\mathcal{B}$ of $\mathcal{A}$, $(\bigvee_{B\in\mathcal{B}} pcl_\gamma(B))(x) < \underline{1}$, for some $x \in X$. Therefore $(\bigwedge_{H\in\mathcal{B}} \overline{pcl_\gamma(H)})(x) > \underline{0}$, for some $x \in X$. Hence $\{\overline{pcl_\gamma(B)} : B \in \mathcal{A}\} = \mathcal{P}$ forms a fuzzy pre-γ-open filter bases in X. Since $\mathcal{A}$ is a pre-γ-open fuzzy set cover of X, we obtain $(\bigwedge_{B\in\mathcal{A}} \overline{B}) = \underline{0}$, which implies $(\bigwedge_{B\in\mathcal{A}} pcl_\gamma(\overline{pcl_\gamma(H)})(x) = \underline{0}$, which is a contradiction. Then each pre-γ-open fuzzy set cover $\mathcal{A}$ of X has a finite subcollection $\mathcal{B}$ such that $(\bigvee_{B\in\mathcal{B}} pcl_\gamma(B)(x) = \underline{1}$, $\forall\ x \in X$. Thus X is fuzzy pre-γ-closed.

Conversely, assume that $\exists$ a fuzzy pre-γ-open filter bases $\mathcal{P}$ in X such that $\bigwedge_{H\in\mathcal{P}} pcl_\gamma(H) = \underline{0}$. Therefore $(\bigvee_{H\in\mathcal{P}} \overline{pcl_\gamma(H)})(x) = \underline{1}$, $\forall\ x \in X$ and thus $\mathcal{A} = \{(\overline{pcl_\gamma(H)}) : H \in \mathcal{A})\}$ is a pre-γ-open fuzzy set cover of X. Since the fts X is fuzzy pre-γ-closed, $\mathcal{A}$ has a finite subcollection $\mathcal{B}$ such that $(\bigvee_{H\in\mathcal{B}} pcl_\gamma(\overline{pcl_\gamma(B)})(x) = \underline{1}$, $\forall\ x \in X$, and thus $\bigwedge_{H\in\mathcal{B}} \overline{(pcl_\gamma(\overline{pcl_\gamma(H)})} = \underline{0}$, Hence $\bigwedge_{H\in\mathcal{B}} H = \underline{0}$, which is a contradiction. Thus $\bigwedge_{H\in\mathcal{P}} pcl_\gamma(H) \neq \underline{0}$.

Theorem 4.2. *A fuzzy subset μ in a fts X is fuzzy pre-γ-closed relative to X iff $\forall$ fuzzy pre-γ-open filter bases $\mathcal{P}$ in X, $(\bigwedge_{\lambda\in\mathcal{P}} pcl_\gamma(\lambda)) \wedge \mu \neq \underline{0}$, $\exists$ a finite subcollection $\mathcal{Q}$ of $\mathcal{P}$ such that $(\bigwedge_{\lambda\in\mathcal{Q}} \lambda)\ \overline{q}\ \mu$.*

Proof. Let μ be a fuzzy pre-γ-closed set relative to X. Suppose $\mathcal{P}$ be a fuzzy pre-γ-open filterbases in X such that for each finite subcollection $\mathcal{Q}$ of $\mathcal{P}$, then $(\bigwedge_{\lambda\in\mathcal{Q}} \lambda)\ q\ \mu$, but $(\bigwedge_{\lambda\in\mathcal{P}} pcl_\gamma(\lambda)) \bigwedge \mu = \underline{0}$. Then $\forall\ x \in S(\mu)$, $(\bigwedge_{\lambda\in\mathcal{P}} pcl_\gamma(\lambda))(x) = \underline{0}$. Therefore $(\bigvee_{\lambda\in\mathcal{P}} \overline{pcl_\gamma(\lambda)})(x) = \underline{1}$, $\forall\ x \in S(\mu)$. So $\mu = \{\overline{pcl_\gamma(\lambda)} : \lambda \in \mathcal{P})\}$ is a pre-γ-open fuzzy set cover of $\mathcal{P}$ and thus $\exists$ a finite sub collection $\mathcal{Q}$ of $\mathcal{P}$, such that $(\bigvee_{\lambda\in\mathcal{Q}} pcl_\gamma(\overline{pcl_\gamma(\lambda)}) \geq \mu$, so that $\bigwedge_{\lambda\in\mathcal{Q}} \overline{(pcl_\gamma(\overline{pcl_\gamma(\lambda)})} = \bigwedge_{\lambda\in\mathcal{Q}} (pint_\gamma(pcl_\gamma(\lambda)) \leq \overline{\mu}$. Thus

$\bigwedge_{\lambda \in \mathcal{Q}} \lambda \leq \overline{\mu}$. Thus $\bigwedge_{\lambda \in \mathcal{Q}} \lambda \, \overline{q} \, \mu$. This is a contradiction.

Conversely, assume that μ is not fuzzy pre-γ-closed relative to X, then $\exists$ a pre-γ-open fuzzy set $\mathcal{A}$ that covers μ such that for each finite subcollection $\mathcal{B}$ of $\mathcal{A}$, we obtain $(\bigvee_{B \in \mathcal{B}} pcl_\gamma(B))(x) \leq \mu(x)$, for some $x \in S(\mu)$ and thus $(\bigwedge_{B \in \mathcal{B}} (\overline{pcl_\gamma(B)})(x) \geq (\overline{\mu(x)}) > \underline{0}$ for some $x \in S(\mu)$. Thus $\mathcal{P} = \{\overline{pcl_\gamma(B)} : B \in \mathcal{A})\}$ forms a fuzzy pre-γ-open filterbases in X. Let $\{\overline{pcl_\gamma(B)} : B \in \mathcal{B})\}$ be a finite subcollection such that $(\bigwedge_{B \in \mathcal{B}} \overline{pcl_\gamma(\lambda)}) \, \overline{q} \, \mu$. Therefore $\mu \leq \bigvee_{B \in \mathcal{B}} pcl_\gamma(B)$. So $\exists$ a finite subcollection $\mathcal{B}$ of $\mathcal{A}$ such that $\mu \leq \bigvee_{B \in \mathcal{B}} pcl_\gamma(B)$, which is a contradiction. Then $\forall$ finite sub collection $\mathcal{Q} = \{(\overline{pcl_\gamma(B)} : B \in \mathcal{B}\}$ of $\mathcal{P}$, we obtain $(\bigwedge_{B \in \mathcal{B}} \overline{pcl_\gamma(\lambda)}) \, q \, \mu$. Therefore by hypothesis $(\bigwedge_{B \in \mathcal{A}} pcl_\gamma(\overline{pcl_\gamma(\lambda)})) \bigwedge \mu \neq \underline{0}$. Therefore $\exists \, x \in S(\mu)$ such that $(\bigwedge_{B \in \mathcal{A}} pcl_\gamma(\overline{pcl_\gamma(\lambda)}))(x) > \underline{0}$. Thus $(\bigvee_{B \in \mathcal{A}} (\overline{pcl_\gamma(\overline{pcl_\gamma(\lambda)})})(x) = (\bigvee_{B \in \mathcal{A}} (pint_\gamma(pcl_\gamma(\lambda))))$ $(x) < \underline{1}$ and thus $(\bigvee_{B \in \mathcal{A}} B) \, (x) < \underline{1}$, which is a contradiction to fact that $\mathcal{A}$ is a pre-γ-open fuzzy set cover of μ. Hence μ is a fuzzy pre-γ-closed relative to X.

Theorem 4.3. *Let $f : (X, \ \tau_X) \to (Y, \ \tau_Y)$ is an onto fuzzy pre*-γ-continuous. If X is fuzzy pre-γ-closed space, then Y is also a fuzzy pre-γ-closed.*

Proof. Let $\{B_\mu : \mu \in \mathcal{P}\}$ be a pre-γ-open fuzzy set cover of Y. Since f is fuzzy pre*-γ-continuous, $\{f^{-1}(B_\mu) : \mu \in \mathcal{P}\}$ is pre-γ-open fuzzy set cover of X. By definition of covering, $\exists$ a finite subcollection $\mathcal{Q}$ of $\mathcal{P}$ such that $(\bigvee_{\mu \in \mathcal{Q}} pcl_\gamma(f^{-1}(B_\mu) = \underline{1}$. Since f is an onto, $\underline{1} = f(\underline{1}) = f(\bigvee_{\mu \in \mathcal{Q}} pcl_\gamma(f^{-1}(B_\mu))) \leq pcl_\gamma(f(f^{-1}(B_\mu)))) = pcl_\gamma(B_\mu)$. Thus Y is fuzzy pre-γ-closed.

5. Fuzzy Pre-γ-Connected Spaces

Definition 5.1. *Let $(X, \ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. Then $(X, \ \tau_X)$ is called fuzzy pre-γ-connected if it has no proper pre-γ-clopen (pre-γ-open and pre-γ-closed) fuzzy subset. [A fuzzy subset μ in X is proper if $\mu \neq \underline{0}$ and $\mu \neq \underline{1}$].*

Theorem 5.1. *A fts $(X, \ \tau_X)$ is fuzzy pre-γ-connected iff it has no non zero pre-γ-open fuzzy subsets λ and μ such that $\lambda + \mu = \underline{1}$.*

Proof. If such λ and μ exist, then λ is a proper fuzzy set which are both pre-γ-open and pre-γ-closed fuzzy set in X. Conversely, if possible assume that X is not fuzzy pre-γ-connected. Then it has a proper fuzzy set λ which are both pre-γ-open and pre-γ-closed fuzzy set. Let us take $\mu = \lambda^c$. Hence μ is a pre-γ-open fuzzy set such that $\mu \neq \underline{0}$ and $\lambda + \mu = \underline{1}$.

Corollary 5.1. *A fts $(X,\ \tau_X)$ is fuzzy pre-γ-connected iff it has no non zero pre-γ-open fuzzy subsets λ and μ such that $\lambda + \mu = \underline{1}$ and $\lambda + \overline{\mu} = \overline{\lambda} + \mu = \underline{1}$.*
Proof. Evident.

Remark 5.1. *The fuzzy product of fuzzy pre-γ-connected spaces may not be a fuzzy pre-γ-connected as shown in the following example.*

Example Let $X_i = [0,1]$, $i \in I$. For some $j,k \in I$, let $\tau_{X_j} = \{0,1,\lambda\}$ and $\tau_{X_k} = \{0,1,\lambda^c\}$, where $\lambda(x) = 1/3$, for $0 \leq x \leq 1$ and $\tau_{X_i} = \{0,1\}$ for each $i \in I$ and $i \neq j,\ i \neq k$. Define an operation γ on τ_{X_j} and τ_{X_k} by $\gamma(\lambda) = \lambda$ and $\gamma(\lambda^c) = \lambda^c$ respectively. Then each X_i is fuzzy pre-γ-connected but $\prod_{i\in I} X_i$ is not so as $\tau(\prod_{i\in I} X_i)$ contains non zero pre-γ-open fuzzy sets $A_j^{-1}(\lambda)$ and $A_k^{-1}(\lambda^c)$ such that for every $x \in \prod_{i\in I} X_i,\ A_j^{-1}(\lambda) + A_k^{-1}(\lambda^c) = 1$. [3]

Definition 5.2. *Let $(X,\ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. A fuzzy subset Y in X is called fuzzy pre-γ-connected if $(Y,\ \tau_Y)$ is a fuzzy pre-γ-connected.*

Theorem 5.2. *Let $(X,\ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. Y is a fuzzy pre-γ-connected subset of X and $\lambda,\ \mu$ are non empty pre-γ-open fuzzy subsets of X such that $\lambda + \mu = \underline{1}_X$, then either $\lambda \wedge Y = \underline{1}_Y$ or $\mu \wedge Y = \underline{1}_Y$.*
Proof. Assume that there exists $x_1,\ x_2 \in Y$ such that $\lambda(x_1) \neq \underline{1}$ and $\mu(X_2) \neq \underline{1}$. Then $\lambda + \mu = \underline{1}$ implies that $\lambda \wedge Y\ +\ \mu \wedge Y = \underline{1}$, where $\lambda \wedge Y \neq \underline{0}$ and $\mu \wedge Y \neq \underline{0}$. Therefore by Theorem 5.1, Y is not a fuzzy pre-γ-connected.

Definition 5.3. *Let $(X,\ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. Two pre-γ-open fuzzy sets λ and μ in X are called fuzzy pre-γ-separated if $\overline{\lambda} + \mu \leq \underline{1}$ and $\lambda + \overline{\mu} \leq \underline{1}$.*

Theorem 5.3. *The image of a fuzzy pre-γ-connected space under a fuzzy pre*-γ-continuous mapping is fuzzy pre-γ-connected.*
Proof. Let $(X,\ \tau_X)$ be a fuzzy pre-γ-connected fts and $f : (X,\ \tau_X) \to (Y,\ \tau_Y)$ be a continuous onto mapping. If possible suppose that $(Y,\ \tau_Y)$ is not fuzzy pre-γ-connected. Then $\exists$ two non empty pre-γ-open fuzzy sets λ and μ of Y such that $\lambda + \mu = \underline{1}$. Therefore, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are two non empty pre-γ-open fuzzy subsets of X such that $f^{-1}(\lambda) + f^{-1}(\mu) = \underline{1}$, which implies X is not fuzzy pre-γ-connected. This is a contradiction.

Definition 5.4. *Let $(X,\ \tau_X)$ be a fts and γ be a fuzzy operation on τ_X. Then X is called fuzzy strongly pre-γ-connected if it has no non zero pre-γ-closed fuzzy sets μ_1 and μ_2 such that $\mu_1 + \mu_2 \leq \underline{1}$. If X is not fuzzy strongly pre-γ-connected then it* *is said to be fuzzy weakly pre-γ-connected.*

Theorem 5.4. *A fts X is fuzzy strongly pre-γ-connected iff it has no non zero pre-γ-open fuzzy sets λ_1 and λ_2 such that $\lambda_1 \neq \underline{1}$, $\lambda_2 \neq \underline{1}$ and $\lambda_1 + \lambda_2 \geq \underline{1}$.*

Proof. Suppose that X is not a fuzzy strongly pre-γ-connected. So X is fuzzy weakly pre-γ-connected.

$\Leftrightarrow$ if it has non zero pre-γ-closed fuzzy sets μ_1 and μ_2 such that $\mu_1 + \mu_2 \leq \underline{1}$.

$\Leftrightarrow$ if it has non zero pre-γ-open fuzzy sets $\lambda = \mu_1^c$ and $\lambda_2 = \mu_2^c$ such that $\lambda_1 \neq \underline{1}$, $\lambda_2 \neq \underline{1}$ and $\lambda_1 + \lambda_2 \geq \underline{1}$.

Remark 5.2. *Every fuzzy strongly pre-γ-connectedness is fuzzy pre-γ-connectedness but the converse need not be true as shown in the following example.*

Example 5.3. Let $X = [0, 1]$ and for $0 \leq x \leq 1$, $\mu(x) = 0.6$ and $\tau_X = \{\underline{1}, \ \underline{0}, \ \mu\}$. Clearly $(X, \ \tau_X)$ is a fts. Define $\gamma : \tau_X \to I^X$ by $\gamma(\underline{1}) = \underline{1}$, $\gamma(\underline{0}) = \underline{0}$, $\gamma(\mu) = \mu$. Then the fts X is a fuzzy pre-γ-connected but not strongly pre-γ-connected.

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